

Tomography of Quantum Spinor States

S. S. Safonov

*Moscow Institute of Physics and Technology,
Institutskii pr. 9, Dolgoprudny, Moscow reg., 141700, Russia
and*

*P. N. Lebedev Physical Institute, Russian Academy of Science,
Leninskii pr. 53, Moscow 117924, Russia*

Abstract

A possibility of describing two-level atom states in terms of positive probability distributions (analog to the symplectic tomography scheme) is considered. As a result the basis of the irreducible representation of a rotation group can be realized by a family of the probability distributions of the spin projection parametrized by points on the sphere. Furthermore the tomography of rotational states of molecules and nuclei which can be described by the model of a symmetric top is discussed.

Recently [1, 2] the symplectic tomography scheme was suggested to obtain the Wigner function by measuring the probability distribution function (the marginal distribution) for a shifted, rotated and squeezed quadrature, which depends on extra parameters. The invertable map of the Wigner function of a quantum state onto the positive marginal distribution for the continuous observable (coordinate) was used to give the formulation of quantum dynamics as the classical statistical process [3, 4, 5]. From this point of view, the Moyal approach [6] to the quantum evolution as to a statistical process was improved in the sense that instead of the Moyal quasidistribution function (the Wigner function) was introduced the positive probability distribution of the measurable variables describing an arbitrary quantum state and its evolution. A spin states reconstruction procedure similar to the symplectic

tomography scheme is investigated in [7, 8]. The authors of these papers showed that in the framework of the symplectic tomography approach it is possible to describe equivalently the spin state in terms of the classical distribution function of a discrete variable instead of the wave function or the density matrix. Taking into account the results of these papers in this article it will be investigated the tomography scheme for two-level atom states and rotational states of molecules and nuclei.

The goal of this work is to make a review of construction of the explicit formula for the invertible map connecting the quantum state of a two-level atom described by the 1/2-spin density matrix $\rho^{(1/2)}$ with the probability distribution function $w(\pm 1/2, \theta, \varphi)$ of the 1/2-spin projection on the quantization axis, where the angles $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ determine the coordinates of the vector normal to the surface of the sphere of the unity radius. On the other hand, quantum characteristics such as rotational energy levels of molecules and nuclei are connected with the tomographic description of the tops' behavior. Therefore in this work it will be also discussed the classical-like description of the quantum states of a top in terms of the distribution functions. For simplicity of presentation, we will consider the example of a symmetric top. These results also are discussed in [9, 10, 11].

1 Tomography of two-level atom states

It is a common knowledge (see, for example [12]) that the wave function $\psi(m)$ of the 1/2-spin particle, which can describe the state of a two-level atom and consists of two components $\psi(1/2)$ and $\psi(-1/2)$, can be represented in the form of a spinor

$$\psi = \begin{pmatrix} \psi(1/2) \\ \psi(-1/2) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \quad (1)$$

In the case of the pure particle's state, the Hermitian density matrix has the form of a 2×2 matrix

$$\rho^{(1/2)} = \psi\psi^\dagger = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}, \quad (2)$$

where the diagonal matrix elements satisfy the normalization condition $|a|^2 + |b|^2 = 1$. Any rotation in the three-dimensional space, determined by

the Euler angles (φ, θ, ψ) varying in the domain $0 \leq \varphi \leq 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$, is associated with a 2×2 unitary matrix

$$u(\varphi, \theta, \psi) = \begin{pmatrix} \cos \frac{\theta}{2} \exp \left[\frac{i(\varphi+\psi)}{2} \right] & \sin \frac{\theta}{2} \exp \left[-\frac{i(\varphi-\psi)}{2} \right] \\ -\sin \frac{\theta}{2} \exp \left[\frac{i(\varphi-\psi)}{2} \right] & \cos \frac{\theta}{2} \exp \left[-\frac{i(\varphi+\psi)}{2} \right] \end{pmatrix}. \quad (3)$$

Under rotation, the Hermitian density matrix $\rho^{(1/2)}$ is transformed as

$$\rho^{(1/2)} \rightarrow \rho^{(1/2)}(u) = u \rho^{(1/2)} u^\dagger, \quad (4)$$

where the 2×2 unitary matrix u is a representation of the angle that describes the matrix elements of the matrix u . For the diagonal elements of the Hermitian density matrix $\rho^{(1/2)}(u)$ we get

$$\rho_{ii}^{(1/2)}(u) = \sum_{s=-1/2}^{1/2} \sum_{m=-1/2}^{1/2} D_{is}^{(1/2)}(u) \rho_{sm}^{(1/2)} D_{im}^{(1/2)*}(u). \quad (5)$$

Here $D_{is}^{(1/2)}$ is the Wigner D-function and $i = -1/2, 1/2$. The diagonal elements of the density matrix of the quantum state take nonnegative values, and their sum is equal to unity. The physical meaning of these elements is that they are a probability to measure the value of the spin projection on the fixed axis in a space. Then, we introduce the notation

$$\rho_{ii}^{(1/2)}(u) = w(i, u), \quad (6)$$

where the function $w(i, u)$ is the marginal distribution, i.e., the probability of finding the spin projection i on the fixed axis in the space for the $1/2$ -spin particle. From Eq. (6), we see that the marginal distribution also depends on the Euler angles φ, θ, ψ as parameters. This distribution function is normalized for all values of the Euler angles. From the structure of Eq. (5), it follows that the marginal distribution $w(i, u)$ depends only on two Euler angles, and does not depend on the angle ψ of the rotation. In fact, according to Eq. (5), one can obtain the marginal distribution $w(i, u)$ of the $1/2$ -spin particle in the following form

$$w(1/2, u) = \cos^2 \frac{\theta}{2} |a|^2 + \frac{\sin \theta}{2} e^{i\varphi} a b^* + \frac{\sin \theta}{2} e^{-i\varphi} b a^* + \sin^2 \frac{\theta}{2} |b|^2 \quad (7)$$

and

$$w(-1/2, u) = \sin^2 \frac{\theta}{2} |a|^2 - \frac{\sin \theta}{2} e^{i\varphi} ab^* - \frac{\sin \theta}{2} e^{-i\varphi} ba^* + \cos^2 \frac{\theta}{2} |b|^2. \quad (8)$$

Hence, the family of the probability distribution functions of the 1/2-spin projection is parametrized by the point's coordinates θ, φ on the sphere of unity radius. This parametrization coincides with the physical meaning of the marginal distribution in the sense that the distribution function $w(i, u)$ is the probability to observe the spin projection i if we measure this spin projection on the quantization axis which is parallel to the vector normal to the surface of the sphere of the unity radius in the point with the coordinates θ and φ . If we know the positive, normalized marginal distribution $w(i, u)$, then, as it was shown in [7, 8], the matrix elements $\rho_{mm'}^{(j)}$ can be calculated with the help of the measurable marginal distribution $w(i, u)$ of the particle with an arbitrary spin j and the values of indices $i = -j, -j+1, \dots, j$ by means of the relation

$$\begin{aligned} (-1)^{m'} \rho_{mm'}^{(j)} &= \sum_{k=0}^{2j} \sum_{l=-k}^k (2k+1)^2 \sum_{i=-j}^j (-1)^i \\ &\otimes \int w(i, u) D_{0l}^k(u) \frac{d\Omega}{8\pi^2} \begin{pmatrix} j & j & k \\ i & -i & 0 \end{pmatrix} \begin{pmatrix} j & j & k \\ m & -m' & l \end{pmatrix} \end{aligned} \quad (9)$$

where $m, m' = -j, -j+1, \dots, j$ and the integration leads over the rotation angles φ, θ, ψ

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_0^\pi \sin \theta d\theta. \quad (10)$$

In the case of the 1/2-spin (i.e., $j = 1/2$), Eq. (9) reads

$$\begin{aligned} \rho_{mm'}^{(1/2)} &= (-1)^{-m'} \sum_{k=0}^1 \sum_{l=-k}^k (2k+1)^2 \sum_{i=-1/2}^{1/2} (-1)^i \\ &\otimes \int w(i, u) D_{0l}^k(u) \frac{d\Omega}{8\pi^2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & k \\ i & -i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & k \\ m & -m' & l \end{pmatrix}, \end{aligned} \quad (11)$$

where $m, m' = -1/2, 1/2$. Calculating all non-zero integrals with D-functions and the marginal distributions and taking into account the expressions of 3j-symbols (see, for example, [12]), we can rewrite Eq. (11) as

follows

$$\begin{aligned}\rho_{mm'}^{(1/2)} = & (-1)^{-m'} \frac{i}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ m & -m' & 0 \end{pmatrix} (|a|^2 + |b|^2) \\ & + \frac{3i}{\sqrt{6}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m & -m' & 0 \end{pmatrix} (|a|^2 - |b|^2) \\ & + i\sqrt{3} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m & -m' & -1 \end{pmatrix} ab^* - i\sqrt{3} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m & -m' & 1 \end{pmatrix} ba^*, \quad (12)\end{aligned}$$

where $m, m' = -1/2, 1/2$, and i is the imaginary unity. The substitution of $m, m' = -1/2, 1/2$ in Eq. (12) leads to the following result:

$$\rho_{mm'}^{(1/2)} = |a|^2, \quad m = m' = 1/2, \quad \rho_{mm'}^{(1/2)} = |b|^2, \quad m = m' = -1/2, \quad (13)$$

and in all the other cases

$$\rho_{mm'}^{(1/2)} = ab^*, \quad m = -m' = 1/2, \quad \rho_{mm'}^{(1/2)} = ba^*, \quad -m = m' = 1/2. \quad (14)$$

Then, we checked by the direct calculation that the right-hand side of Eq. (11) is equal to the density matrix (2). One can conclude, that given a measurable marginal distribution of a particle, whose state is described in terms of spinor, one can reconstruct the state density matrix by means of Eq. (11).

2 Examples of the marginal distribution for the 1/2 and 1 spin states

The classical-like description of quantum mechanics in terms of the positive, normalized marginal distribution can be easily understood in the case of the 1/2-spin particle. It is a common knowledge [12], that the spin operator of the 1/2-spin particle has the form

$$\hat{s} = \frac{1}{2}\hat{\sigma}, \quad (15)$$

where the Pauli matrixes are

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (16)$$

The wave function of the 1/2-spin particle can be represented in the form of spinor (1). When the direction of the particle's spin coincided with the positive direction of the x axis in the coordinate system (x, y, z) , the wave function of spinor has the form

$$\psi_x^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (17)$$

And in the case of a contrary direction of the spin with respect to the x axis, the wave function reads

$$\psi_x^{(-)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (18)$$

Analogously for the y -projection, when the direction of the particle's spin is $+1/2$ or $-1/2$ on the y and z axes, we have

$$\psi_y^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad (19)$$

$$\psi_y^{(-)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad (20)$$

and for the z -projection

$$\psi_z^{(+)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (21)$$

$$\psi_z^{(-)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (22)$$

For all values of the 1/2-spin directions on the x , y and z axes the density matrices have the following forms

$$\rho_x^{(+)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho_x^{(-)} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad (23)$$

$$\rho_y^{(+)} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \rho_y^{(-)} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad (24)$$

and

$$\rho_z^{(+)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_z^{(-)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (25)$$

Applying Eqs. (7) and (8), we obtain the marginal distributions for the different values of the 1/2-spin directions on the x , y and z axes

$$w_x^{(+)}(+1/2, u) = \frac{1}{2}(1 + \sin \theta \cos \varphi), \quad w_x^{(+)}(-1/2, u) = \frac{1}{2}(1 - \sin \theta \cos \varphi) \quad (26)$$

for the positive x -projection,

$$w_x^{(-)}(+1/2, u) = \frac{1}{2}(1 - \sin \theta \cos \varphi), \quad w_x^{(-)}(-1/2, u) = \frac{1}{2}(1 + \sin \theta \cos \varphi) \quad (27)$$

for the negative x -projection,

$$w_y^{(+)}(+1/2, u) = \frac{1}{2}(1 + \sin \theta \sin \varphi), \quad w_y^{(+)}(-1/2, u) = \frac{1}{2}(1 - \sin \theta \sin \varphi) \quad (28)$$

for the positive y -projection,

$$w_y^{(-)}(+1/2, u) = \frac{1}{2}(1 - \sin \theta \sin \varphi), \quad w_y^{(-)}(-1/2, u) = \frac{1}{2}(1 + \sin \theta \sin \varphi) \quad (29)$$

for the negative y -projection,

$$w_z^{(+)}(+1/2, u) = \cos^2 \frac{\theta}{2}, \quad w_z^{(+)}(-1/2, u) = \sin^2 \frac{\theta}{2} \quad (30)$$

for the positive z -projection and

$$w_z^{(-)}(+1/2, u) = \sin^2 \frac{\theta}{2}, \quad w_z^{(-)}(-1/2, u) = \cos^2 \frac{\theta}{2} \quad (31)$$

for the negative z -projection. In Figs. 1–2, we plot the marginal distributions of the 1/2-spin particle $w_x^{(\pm)}(\pm 1/2, \theta, \varphi)$ the spin direction of which coincides with the positive direction of the x axis and the spin projection onto a fixed axis in space is equal to $\pm 1/2$, as a function of the angles θ and φ .

However, there exist the mixed states of the 1/2-spin particle (see, for example, [12]). These states are described only by the density matrix

$$\rho_m = \begin{pmatrix} \frac{1}{2} + \bar{s}_z & \bar{s}_- \\ \bar{s}_+ & \frac{1}{2} - \bar{s}_z \end{pmatrix}. \quad (32)$$

In Eq. (32), \bar{s}_{\pm} are determined by the relation $\bar{s}_{\pm} = \bar{s}_x \pm i\bar{s}_y$, and \bar{s}_x , \bar{s}_y and \bar{s}_z are the mean values of the 1/2-spin projections on the x , y and z axes,

respectively. These parameters satisfy the condition $\bar{s}_x^2 + \bar{s}_y^2 + \bar{s}_z^2 \leq 1/4$. A parameter $\mu = \text{Tr}(\rho_m^2)$ is named “purity state degree” and depends on the mean values of the spin projection $\mu = 1/2 + 2(\bar{s}_x^2 + \bar{s}_y^2 + \bar{s}_z^2)$. As in the case of a pure state, using Eq. (5) for the matrix elements of the density matrix of a mixed state, we calculate the marginal distribution of the mixed state of the 1/2-spin particle,

$$w_m\left(\frac{1}{2}, u\right) = \frac{1}{2} + \bar{s}_z \cos \theta + \bar{s}_x \sin \theta \cos \varphi + \bar{s}_y \sin \theta \sin \varphi \quad (33)$$

and

$$w_m\left(-\frac{1}{2}, u\right) = \frac{1}{2} - \bar{s}_z \cos \theta - \bar{s}_x \sin \theta \cos \varphi - \bar{s}_y \sin \theta \sin \varphi. \quad (34)$$

It is easy to check that the examples of the pure-state marginal distribution (26)-(31) are obtained from Eqs. (33), (34) by the proper choice of the parameters \bar{s}_x , \bar{s}_y and \bar{s}_z .

In this section, one can also consider another one important example of the 1-spin particle states, the direction of which coincides with the positive direction of the z axis. In this particular case, the wave function has the form

$$\psi_z^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (35)$$

and the density matrix, consequently, takes the form

$$\rho^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (36)$$

The marginal distribution of the spin state can be calculated by the following equation (see, for example, [12])

$$\rho_{ii}^{(1)}(u) = D_{si}^1(u) \rho_{sm}^{(1)} D_{mi}^{1*}(u), \quad (37)$$

where u is determined by Eq. (10). As a result, we obtain the marginal distribution $w(i, \varphi, \theta, \psi) \equiv \rho_{ii}^{(1)}(u)$, $\sum_{i=-1}^1 w(i, \phi, \theta, \varphi) = 1$, i.e., three probabilities for the 1-spin projection onto the z axes (-1 , 0 , and $+1$), which also depend on the rotation angles of the reference frame φ , θ , and ψ

$$w(1, \theta) = \frac{(1 + \cos \theta)^2}{4}, \quad (38)$$

$$w(0, \theta) = \frac{(1 - \cos^2 \theta)}{2}, \quad (39)$$

$$w(-1, \theta) = \frac{(1 - \cos \theta)^2}{4}. \quad (40)$$

A dependence on the ϕ, φ angles drops out. Substituting this marginal distribution into formula (9) by means of which matrix elements of the density matrix $\rho_{mm'}^{(j)}$ can be reconstructed by the measurable marginal distribution $w(i, \theta)$, $i = -1, 0, 1$ and executing the calculations, we could verify the correction of this formula in the particular case of the unity spin. In the case $j = 1$, this formula consists of twenty seven items. Let us consider all values of the Wigner D-functions for the various values of their subscripts and superscripts. Because the marginal distribution $w(i, \theta)$, $i = -1, 0, 1$ takes only positive values, and because some D-functions contain a complex-valued exponential coefficient, integration of the product of the marginal distribution and a D-function over the ϕ, θ, φ rotation variables yields zero. Taking this fact into account, Eq. (9) can be simplified, and we arrive at

$$\begin{aligned} \rho_{mm'}^{(1)} = & \frac{(-1)^{m'}}{8\pi^2} \left[\sum_{i=-1}^1 \int d\Omega (-1)^i w(i, \theta) D_{00}^0 \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ m & -m' & 0 \end{pmatrix} \right. \\ & + 9 \sum_{i=-1}^1 \int d\Omega (-1)^i w(i, \theta) D_{00}^1 \begin{pmatrix} 1 & 1 & 1 \\ i & -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ m & -m' & 0 \end{pmatrix} \\ & \left. + 25 \sum_{i=-1}^1 \int d\Omega (-1)^i w(i, \theta) D_{00}^2 \begin{pmatrix} 1 & 1 & 2 \\ i & -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ m & -m' & 0 \end{pmatrix} \right]. \quad (41) \end{aligned}$$

Calculating all nonzero integrals and taking into account the expressions of the $3j$ -symbols from [12] we can rewrite Eq. (41) into the form

$$\begin{aligned} \rho_{mm'}^{(1)} = & (-1)^{-m'+1} \left[\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 0 \\ m & -m' & 0 \end{pmatrix} \right. \\ & \left. + \frac{3}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ m & -m' & 0 \end{pmatrix} + \frac{5}{\sqrt{15}} \begin{pmatrix} 1 & 1 & 2 \\ m & -m' & 0 \end{pmatrix} \right]. \quad (42) \end{aligned}$$

For $m \neq m'$ ($m, m' = -1, 0, 1$), $3j$ -symbols are equal to zero (see, for example, [12]). Evaluating all the remaining $3j$ -symbols for $m = m'$, we obtain

$$\rho_{mm'}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m, m' = -1, 0, 1. \quad (43)$$

This means that we demonstrated how the initial density matrix (36) can be recovered, if we know the marginal distribution of the quantum state (38)-(40).

3 Tomography of rotational states of molecules and nuclei

Here we will consider the tomography scheme of rotational states of molecules and nuclei using, for simplicity, the model of a symmetric top. It is a common knowledge (see, for example, [12]) that the Hamiltonian of a symmetric top has the form

$$\widehat{H} = \frac{\hbar^2}{2I_A} \widehat{J}^2 + \frac{\hbar^2}{2} \left(\frac{1}{I_C} + \frac{1}{I_A} \right) \widehat{J}_\zeta^2, \quad (44)$$

where I_A , I_C are the principal inertia momenta of a top (two of the momenta of a symmetric top coincide with each other), and \widehat{J} is the angular-momentum operator. Stationary states of a symmetric top are characterized by three quantum numbers: the orbital momentum j and its projections onto the top axis ($J_\zeta = k, k = -j, -j+1 \dots j$) and onto a fixed z axis in space ($J_z = M, M = -j, -j+1 \dots j$). The symmetric top energy is independent of the last quantum number M . Let us consider the stationary states of the symmetric top with given energy. To do this, we take a subspace of the $(2j+1)^2$ dimensions with the fixed j in the state space. The wave function of the stationary state of a symmetric top can be represented in the form

$$\psi_{Mk}^{(j)} = \langle jMk | \psi \rangle = \psi_{jk}^{(0)} D_{kM}^j, \quad (45)$$

where D_{kM}^j is the Wigner D-function, and $\psi_{jk}^{(0)}$ is the wave function in the reference frame, which comoving with the physical system (the top). For a pure state, the density matrix of the symmetric top is expressed in terms of the wave functions as follows

$$\rho_{MkM'k'}^{(j)} = \psi_{Mk}^{(j)} \psi_{M'k'}^{(j)*}. \quad (46)$$

Thus, under two consecutive rotations (these rotations are determined by the Euler angles $u(\phi, \theta, \varphi)$ and $u'(\varphi', \theta', \psi')$) the Hermitian density matrix ρ is transformed as

$$\rho \rightarrow \rho^{(j)}(u, u') = D^{(j)}(u') \rho D^{(j)}(u) D^{(j)\dagger}(u) D^{(j)\dagger}(u'), \quad (47)$$

where unitary $(2j+1) \times (2j+1)$ matrixes $D(u)$ and $D(u')$ represent these two rotations through the angles u and u' . Since the density matrix under consideration depends on four discrete indices, it is necessary to do two consecutive rotations of the reference frame, which comoving with the top to obtain the diagonal elements of the transformed density matrix. Taking into account this fact, for the diagonal elements of the density matrix $\rho^{(j)}(u, u')$ can be reduced to the form

$$\rho_{i_1 i_2 i_1 i_2}^{(j)}(u, u') = D_{pi_2}^j(u') D_{ni_1}^j(u) \rho_{npsl}^{(j)} D_{si_1}^{j*}(u) D_{li_2}^{j*}(u'). \quad (48)$$

Here $D_{ni_1}^j(u)$ is the above Wigner D-function and $i_1, i_2, n, p, s, l = -j, -j+1, \dots, j$. In Eq. (48), we assume that summation is performed over the repeated indices n, p, s, l , but there is no summation over the indices i_1, i_2 . Below, we will assume that the Hermitian nonnegative density matrix $\rho_{npsl}^{(j)}$ describes not only the pure state of the top (45) but also an arbitrary mixed state, i.e. $Tr(\rho^{(j)})^2 < 1$.

Let us discuss the problem of reconstructing all matrix elements $\rho_{npsl}^{(j)}$, if the diagonal elements $\rho_{i_1 i_2 i_1 i_2}^{(j)}(u, u')$ are known. To do this, let us multiply both sides of Eq. (48) by the functions $D_{ab}^c(u)$ and $D_{de}^f(u')$ with arbitrary upper indices c, f , and with different angular variables u and u' . As a result, we have

$$\begin{aligned} \rho_{i_1 i_2 i_1 i_2}^{(j)}(u, u') D_{ab}^c(u) D_{de}^f(u') &= D_{pi_2}^j(u') D_{ni_1}^j(u) \rho_{npsl}^{(j)} D_{si_1}^{j*}(u) D_{li_2}^{j*}(u') \\ &\quad \otimes D_{ab}^c(u) D_{de}^f(u'). \end{aligned} \quad (49)$$

Integrating over the angular variables $u(\phi, \theta, \varphi)$ and $u'(\phi', \theta', \varphi')$,

$$\int d\Omega = \frac{1}{8} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi, \quad (50)$$

$$\int d\Omega' = \frac{1}{8} \int_0^{2\pi} d\phi' \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} d\varphi', \quad (51)$$

using the well-known expression of the integral of the three D-functions product [12] and

$$D_{m'm}^{j*}(u) = (-1)^{m'-m} D_{-m', -m}^j(u), \quad (52)$$

along with the orthogonality properties of the $3j$ -symbols, one can express $\rho_{npsl}^{(j)}$ in terms of the diagonal elements $\rho_{i_1 i_2 i_1 i_2}^{(j)}(u, u')$. As it was mentioned above, the diagonal elements of the density matrix for a quantum state take nonnegative values and their sum is equal to unity. Thus, we introduce the notation $\rho_{i_1 i_2 i_1 i_2}^{(j)}(u, u') = w(i_1, i_2, u, u')$, where the function $w(i_1, i_2, u, u')$ is the marginal distribution of a symmetric top, i.e. the probability of finding the projection i_1 of the angular momentum onto a fixed axis in space and the projection i_2 of the angular momentum onto the top axis, and also this probability parametrically depends on Euler angles $\phi, \theta, \psi, \phi', \theta', \psi'$. This function is normalized $\sum_{i_1, i_2=-j}^j w(i_1, i_2, u, u') = 1$ for all values of the Euler angles. Suppose we know the positive, normalized marginal distribution $w(i_1, i_2, u, u')$. Thus, in view of Eqs. (49)-(52) the matrix elements $\rho_{m_1 m_2 m'_1 m'_2}^{(j)}$ can be expressed in terms of the measurable marginal distribution $w(i_1, i_2, u, u')$ with $i_1, i_2 = -j, -j+1, \dots, j$ by using the relationship

$$\begin{aligned} \rho_{m_1 m_2 m'_1 m'_2}^{(j)} &= \sum_{k_1=0}^{2j} \sum_{l_1=-k_1}^{k_1} (2k_1+1)^2 \sum_{k_2=0}^{2j} \sum_{l_2=-k_2}^{k_2} (2k_2+1)^2 \\ &\otimes \sum_{i_1=-j}^j \sum_{i_2=-j}^j (-1)^{i_1+i_2-m'_1-m'_2} \int \int w(i_1, i_2, u, u') D_{0l_1}^{k_1}(u) \frac{d\Omega}{8\pi^2} \\ &\otimes D_{0l_2}^{k_2}(u') \frac{d\Omega'}{8\pi^2} \begin{pmatrix} j & j & k_1 \\ i_1 & -i_1 & 0 \end{pmatrix} \begin{pmatrix} j & j & k_1 \\ m_1 & -m'_1 & l_1 \end{pmatrix} \\ &\otimes \begin{pmatrix} j & j & k_2 \\ i_2 & -i_2 & 0 \end{pmatrix} \begin{pmatrix} j & j & k_2 \\ m_2 & -m'_2 & l_2 \end{pmatrix}. \end{aligned} \quad (53)$$

Here $m_1, m_2, m'_1, m'_2 = -j, -j+1, \dots, j$. One can conclude that given a measurable marginal distribution of the symmetric top, one can reconstruct the stationary-state density matrix of the symmetric top by means of Eq. (53).

References

- [1] S. Mancini, V. I. Man'ko and P. Tombesi, *Quantum Semiclass. Opt.*, **7**, 615, (1995).
- [2] G. M. D'Ariano, S. Mancini, V. I. Man'ko and P. Tombesi, *Quantum Semiclass. Opt.*, **6**, 1017, (1996).

- [3] S. Mancini, V. I. Man'ko and P. Tombesi, *Phys. Lett. A.*, **213**, 1, (1996);
“Classical-like description of quantum dynamics by means of symplectic tomography”, Los Alamos Preprint quant-ph/9609026.
- [4] S. Mancini, V. I. Man'ko and P. Tombesi, *Found. Phys.*, **27**, 801, (1997).
- [5] V. I. Man'ko, “Quantum mechanics and classical probability theory,”
in Proceedings of International Conference “Symmetries in Science IX”,
215, (Bregenz, Austria, August 1996).
- [6] J. E. Moyal, *Proc. Cambridge Philos. Soc.*, **45**, 99, (1949).
- [7] V. V. Dodonov and V. I. Man'ko, *Phys. Lett. A.*, **229**, 335, (1997).
- [8] V. I. Man'ko and O. V. Man'ko, *JEFT*, **112**, 796, (1997).
- [9] O. V. Man'ko, V. I. Man'ko, and S. S. Safonov, *TMPH*, **115**, 2, 185,
(1998).
- [10] V. I. Man'ko and S. S. Safonov, *Physics of Atomic Nuclei*, **61**, 4, 585,
(1998).
- [11] V. A. Andreev, O. V. Man'ko, V. I. Man'ko, and S. S. Safonov, *Journal
of Russian Laser Research*, (to be published, 1998).
- [12] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon, New
York, (1958).

Figure 1

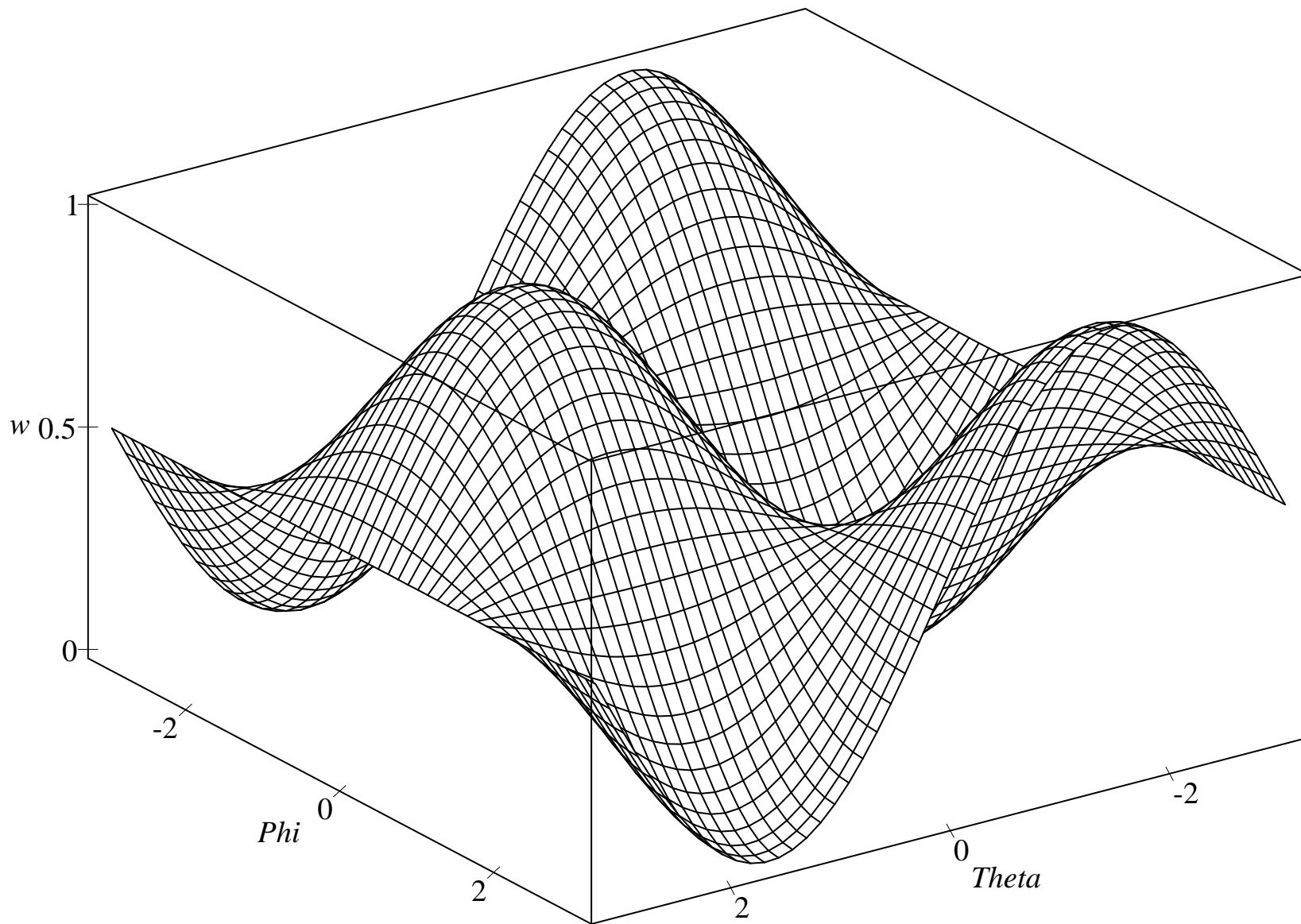


Figure 2

